KINGDOM OF SAUDI ARABIA
MINISTRY OF EDUCATION
ALBAHA UNIVERSITY

July - December 2019 Volume 3 Issue 2

ALBAHA UNIVERSITY JOURNAL OF
BASIC AND APPLIED
SCIENCES

Hopf Bifurcation on Fractional Ordered Glucose-Insulin System with Time-Delay

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p-ISSN:1658-7529
e-ISSN:1658-7537

A Refereed Academic Journal Published by Albaha University

ALBAHA UNIVERSITY JOURNAL OF

BASIC AND APPLIED SCIENCES

July - December 2019 Volume 3 Issue 2

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http://bu.edu.sa
Published by BUJBAS, Albaha University,
65451 Albaha, Kingdom of Saudi Arabia

All scientific articles in this issue are refereed.

SCOPE

Albaha University Journal of Basic and Applied Sciences (BUJBAS) publishes English language, peer-reviewed papers focused on the integration of all areas of sciences and their application. Supporting the concept of interdisciplinary BUJBAS welcomes submissions in various academic areas such as medicine, dentistry, pharmacy, biology, agriculture, veterinary medicine, chemistry, mathematics, physics, engineering, computer sciences and geology.

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BUJBAS is published by Albaha University. For queries related to the journal, please contact
http://sj.bu.edu.sa
E-mail: bujs@bu.edu.sa

L.D. No: 1438/2732

p-ISSN: 1658-7529

e-ISSN: 1658-7537

Albaha University Journal of Basic and Applied Sciences 3(2) (2019) 27–34

Article available at Albaha University Journal of Basic and Applied Sciences

Journal of Basic and Applied Sciences

Journal homepage: http://sj.bu.edu.sa/

Hopf Bifurcation on Fractional Ordered Glucose-Insulin System with Time-Delay

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ARTICLEINFO

Article history:
Received 8 April 2019
Received in revised form
7 September 2019
Accepted 15 September 2019

Keywords:
Caputo's derivative,
Laplace transformation method

ABSTRACT

This paper considers a class of fractional-order glucose-insulin interaction with time delay for analyzing the dynamic behaviors such as Hopf bifurcation, local asymptotic stability and global asymptotic stability. The stability of the equilibrium state is investigated by analyzing the eigenvalue of the corresponding characteristic matrix for the fractional-order time delay models using a Laplace transformation for the Caputo-fractional derivatives. Some sufficient conditions are established to guarantee the uniqueness of the equilibrium point. Numerical simulations have been used to verify the theoretical analysis.

1. Introduction

This paper is a continuation of [1] and [2]. Stability theory is one of the most important and rapidly developing fields of applied mathematics and mechanics. The stability studies are well known. The dynamic behaviors such as periodic phenomenon, bifurcation and chaos are also of great interest. Diabetes Mellitus is a disease, which characterized by too high sugar levels in the blood and urine. According to the International Diabetes Federation (IDF), 387 million people have diabetes in 2014, and this number will rise to 592 million by 2035. Diabetes also caused 4.9 million deaths in 2014; every seven seconds a person dies from diabetes (http://www.idf.org/diabetesatlas/update-2014).

The model for the interaction of glucose and insulin based on the work of [3] is discussed by Hussain-Zadeng in [4].

In this paper we investigate the stability and Hopf bifurcation of the fractional-order for the interaction of glucose and insulin model with time delays. We develop a Hopf bifurcation theory for a system of fractional differential equations with time delay. Assume that τ is the time delay that represents the time taken by Pancreas to respond to the feedback of the glucose level. It is used as the bifurcation parameter. Here, we analyses the dynamical behavior of the extended system of Hussain-Zadeng in [3]:

$$\begin{aligned} D^\alpha x(t) &= -a_1 x(t) - a_2 x(t) y(t) + a_3, t \in [0,T] \\ D^\alpha y(t) &= b_1 x(t-\tau) - b_2 y(t), t \in [0,T] \end{aligned} \tag{1}$$

with

$$x(\theta) = \phi(\theta), \theta < 0,$$
$$y(\theta) = \psi(\theta), \theta < 0,$$

where $x \geq 0$, $y \geq 0$, represents glucose and insulin concentration respectively, a_1 and a_2 are the rate constants which represent insulin-independent and insulin-dependent glucose disappearance respectively, a_3 is the glucose infusion rate and b_1 is the rate constant which represents insulin production due to glucose stimulation, b_2 is the rate constant which represents insulin degradation.

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The fractional calculus has a long history, starting from 1695 when the derivative of order $\alpha = 1/2$ was described by Leibniz [5]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grunwald, Letnikov and Riemann. Derivatives and integrals of fractional order have found many applications in recent studies in engineering mathematics, applied mathematics, and many fields of physics [6, 7].

Broad classes of analytical methods have been proposed for solving fractional differential equations, such as the decomposition methods [8, 9], variational iteration methods [10-14], differential transform methods [15] and the homotopy perturbation method [16-18].

The sufficient conditions for stability of the fractional system with respect to the equilibrium point of the system are derived by using some inequality techniques. By using the

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bifurcation point and critical frequency the transversally condition is verified for the point of bifurcation occurring. Numerical simulations have been used to verify the theoretical analysis.

There are three main definitions of fractional-order differential, that is, Grunwald-Letnikov, Riemann-Liouville and Caputo's definitions. Here, this paper is based on the Caputo definition.

Let $\Gamma(z) = \int_0^\infty e^{-t} e^{z-1} dt$ be the Euler gamma function. Following [7], the Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in N$ is defined as:

$$D^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx, n-1 < \alpha < n, \\ \dfrac{d^n}{dt^n} f(t), \qquad \alpha = n. \end{cases}$$

2. Stability Analysis

This section deals with investigation of the stability and Hopf bifurcation of the delayed fractional-order model (1) for $\alpha = 1$. More precisely, for $t \in [0, T]$, we study the following model:

$$\begin{aligned} \frac{dx}{dt} &= -a_1 x(t) - a_2 x(t) y(t) + a_3, \\ \frac{dy}{dt} &= b_1 x(t-\tau) - b_2 y(t). \end{aligned} \tag{2}$$

The steady state of the system (2) is one of which:

$x(t) = x(t-\tau)$ and $D^\alpha x(t) = 0$,

$y(t) = y(t-\tau)$ and $D^\alpha y(t) = 0$.

It is easy to see that system (2) has the equilibrium point $E_1 = (x^*, y^*)$ as the following:

$$x^* = \frac{-a_1 b_2 + \sqrt{(a_1 b_2)^2 + 4 a_2 b_2 a_3 b_1}}{2 b_1 a_2},$$
$$y^* = \frac{-a_1 b_2 + \sqrt{(a_1 b_2)^2 + 4 a_2 b_2 a_3 b_1}}{2 b_2 a_2}.$$

By using $G(t) = x(t) - x^*$, $I(t) = y(t) - y^*$, the linear model of system (1) about E_1 is given by:

$$\begin{aligned} DG(t) &= -a_1 G(t) - a_2 y^* G(t) - a_2 x^* I(t), \\ DI(t) &= b_1 G(t-\tau) - b_2 I(t). \end{aligned} \tag{3}$$

Using Laplace transform [19] on both sides of (3), one obtains

$$\lambda^2 + A\lambda + B + C e^{-\lambda\tau} = 0, \tag{4}$$

where $A = a_1 + a_2 y^* + b_2$, $B = a_1 b_2 + a_2 b_2 y^*$, and $C = b_1 a_2 x^*$.

Lemma 1. If $B + C \neq 0$ and $B^2 > C^2$, then the number of pairs of pure imaginary roots of the characteristic equation (4) is zero for $\tau > 0$. (i.e., the unique equilibrium point of (4) is stable for all $\tau > 0$).

Proof. We observed that $\lambda = 0$ cannot be a root of the characteristic equation (4), since $B + C \neq 0$. Therefore, the only possibility for instability is Hopf bifurcation. We have to check whether there exists a real $\mu > 0$, so that $\lambda = i\mu$ is a root of:

$$\Delta(\lambda) = 0.$$

Here, $\Delta(\lambda)$ is considered as a characteristic matrix of system (2). Substituting $\lambda = i\mu$ in (4), one obtains

$$-\mu^2 + Ai\mu + B + Ce^{-i\mu\tau} = 0. \tag{5}$$

Equation (5) becomes $Ce^{-i\mu\tau} = \mu^2 - Ai\mu - B$.

From this, one obtains,

$$\left|\frac{\mu^2 - Ai\mu - B}{C}\right| \leq 1. \tag{6}$$

By choosing $G(\mu) = (\mu^2 - B)^2 + A^2\mu^2 C^2$ and $G(0) = \left(\frac{B}{C}\right)^2$, we have

$$\frac{dG(\mu)}{d\mu} = 4\mu(\mu^2 - B) + 2A^2 \mu C^2 \geq 0.$$

This shows that $G(\mu)$ is an increasing function of μ and $G(0) > 1$, it follows that $G(\mu) > 1$ for all $\mu \geq 0$. This contradicts Eq. (6).

From Lemma 1, we conclude that there is no stability switch. So the system is always locally asymptotically stable.

Now, we examine the stability of system (2) for $\tau = 0$, the characteristic equation (4) becomes

$$\Delta(\lambda) = \lambda^2 + A\lambda + B + C = 0. \tag{7}$$

Eq. (7) can be converted to

$$\lambda^2 + D_1\lambda + D_2 = 0,$$

where $D_1 = A$, $D_2 = B + C$. Let us define

$$\psi_1 = D_1, \psi_2 = D_1 D_2,$$

Then from the above results, we make the following hypotheses: $\psi_i > 0$ $(i = 1,2)$.

Lemma 2. If $\psi_1 > 0$ and $\psi_2 > 0$ holds, then the zero equilibrium point of the fractional order system (2) is asymptotically stable when $\tau = 0$.

Remark 1. The conditions $\psi_1 > 0$ and $\psi_2 > 0$ are sufficient condition for Lemma 2. If the conditions are retrieved by another method which entails that all the roots of equation (1) satisfy

$$|\arg(\lambda)| > \alpha\pi/2,$$

then Lemma 2 may still hold.

Lemma 3. If $B < C$, then the transversality conditions

$$\left.\frac{d(Re\,\lambda)}{d\tau}\right|_{\tau=\tau_0, \mu=\mu_0} > 0 \tag{8}$$

hold.

Proof. Differentiating the both sides of Eq. (4) with respect to τ and noticing that λ is a function of τ, one obtains

$$[2\lambda + A - C\tau e^{-\tau\lambda}] \frac{d\lambda}{d\tau} = C\lambda e^{-\tau\lambda},$$

which implies

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda+A)\, e^{\tau\lambda}}{C\lambda} - \frac{\tau}{\lambda}.$$

Noting that $\lambda = \pm i\mu_0$ when $\mu = \mu_0$ satisfy (6), therefore, one obtains

$$\left(\frac{d(Re\,\lambda)}{d\tau}\right)^{-1}_{\tau=\tau_0,\mu=\mu_0} = Re\left[\frac{(2\lambda+A)\, e^{\tau\lambda}}{C\lambda}\right]_{\tau=\tau_0,\mu=\mu_0} = \frac{1}{C^2}\sqrt{(A^2-4B)A^2+4C^2} > 0.$$

Thus, the transversability condition

$$\left.\frac{d(Re\,\lambda)}{d\tau}\right|_{\tau=\tau_0,\mu=\mu_0} > 0$$

holds, and hence Hopf bifurcation occurs at $\tau = \tau_0$. This completes the proof.

From Lemma 3, we have the following result.

Lemma 4. If $\tau \neq \tau_0$, then Eq. (4) has at least one root with strictly positive real part.

By Lemmas 1-4, we have the following result on stability and bifurcation of system (1).

Lemma 5. If $q \geq r$, then the positive equilibrium E_1 of system (1) is asymptotically stable for any $\tau \geq 0$ if $q < r$, then the positive equilibrium E_1 of system (1) is asymptotically stable when $t \in [-\tau, 0]$.

Remark 2. System (1) undergoes a Hopf bifurcation at the origin when $\tau = 0$.

Remark 3. Lemma 5 implies that the transversality condition (8) of Hopf bifurcations is satisfied for the delayed fractional-order model (1).

3. Local and global stability analysis for fractional system

This section deals with investigation of the stability and bifurcation of the model (1). The interior-equilibrium point $E_1 = (x^*, y^*)$ exists unconditionally as x^* and y^* are always positive as all the parameters are considered positive. To linearize the model (1) about E_1, let $G(t) = x(t) - x^*$ and $I(t) = y(t) - y^*$. After removing nonlinear terms, one obtains

$$\begin{aligned} D^\alpha G(t) &= -a_1 G(t) - a_2 y^* G(t) - a_2 x^* I(t), \\ D^\alpha I(t) &= b_1 G(t-\tau) - b_2 I(t), 0 < \alpha \leq 1. \end{aligned} \tag{9}$$

Taking Laplace transform [19] on both sides of (9), one obtains the associated characteristic equation as follows:

$$\begin{vmatrix} \lambda^\alpha + a_1 + a_2 y^* & a_2 x^* \\ -b_1 e^{-\lambda\tau} & \lambda^\alpha + b_2 \end{vmatrix} = 0.$$

Thus the characteristic equation is:

$$\lambda^{2\alpha} + A\lambda^\alpha + B + Ce^{-\lambda\tau} = 0, \tag{10}$$

where $A = a_1 + a_2 y^* + b_2$, $B = a_1 b_2 + a_2 b_2 y^*$, and $C = a_2 b_1 x^*$. Equation (10) can be rewritten in the following characteristic polynomial:

$$P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau} = 0, \tag{11}$$

where $P_1(\lambda) = \lambda^{2\alpha} + A\lambda^\alpha + B$ and $P_2(\lambda) = C$. Multiplying both sides of (11) by $e^{\lambda\tau}$, we get

$$P_1(\lambda)e^{\lambda\tau} + P_2(\lambda) = 0. \tag{12}$$

Let A_r and B_r be the real and imaginary parts of $P_r(\lambda)$ $(r = 1, 2)$ and be defined as

$$\begin{aligned} A_1 &= \mu^{2\alpha}\cos\alpha\pi + C_{11}\mu^\alpha \cos\frac{\alpha\pi}{2} - C_{12}\gamma^\alpha \sin\frac{\alpha\pi}{2} + C_{21} \\ A_2 &= C_{31}\mu^\alpha \cos\frac{\alpha\pi}{2} - C_{32}\gamma^\alpha \sin\frac{\alpha\pi}{2}, \\ B_1 &= \mu^{2\alpha}\sin\alpha\pi + C_{11}\mu^\alpha \sin\frac{\alpha\pi}{2} + C_{12}\gamma^\alpha \cos\frac{\alpha\pi}{2} + C_{22} \\ B_2 &= C_{31}\mu^\alpha \sin\frac{\alpha\pi}{2} + C_{32}\gamma^\alpha \cos\frac{\alpha\pi}{2}, \end{aligned} \tag{13}$$

where $C_{11} = Re\,(A)$, $C_{21} = Re\,(B)$, $C_{31} = Re\,(C)$, $C_{12} = Im\,(A)$, $C_{22} = \text{Im}\,(B)$ and $C_{32} = \text{Im}\,(C)$. Then, by using A_r and B_r and substituting (13) in (12), we get

$$(A_1 + iB_1)e^{\lambda\tau} + (A_2 + iB_2) = 0. \tag{14}$$

Further, we will analyze stability and bifurcation properties; for this we consider the real number $\mu > 0$, there exists $\lambda = i\mu = \mu\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, then we substitute the expression of λ into (14) and separating the real and imaginary parts of it, it results in

$$\begin{aligned} A_1\cos\mu\tau - B_1\sin\mu\tau &= -A_2, \\ A_1\cos\mu\tau - B_1\sin\mu\tau &= -B_2, \end{aligned} \tag{15}$$

Utilizing Eq. (15), direct calculation yields

$$\cos\mu\tau = -\frac{A_1A_2 + B_1B_2}{A_2^2 + B_2^2} = g_1(\mu),$$
$$\sin\mu\tau = \frac{A_2B_1 - A_1B_2}{A_2^2 + B_2^2} = g_2(\mu).$$

It is clear that

$$g_1^2(\mu) + g_2^2(\mu) = 1.$$

Thus, it follows from $\cos\mu\tau = g_1(\mu)$ that

$$\tau^{(k)} = \frac{1}{\mu}(\arccos g_1(\mu) - 2k\pi), k = 0, 1, 2, \ldots \tag{16}$$

We suppose that Eq. (14) has at least one positive real root. Define the bifurcation point:

$$\tau_0 = \min\{\tau^{(k)}\}, k = 0, 1, 2, \ldots$$

where $\tau^{(k)}$ is defined by (16).

In order to achieve the transversality condition for the occurrence for Hopf bifurcation, the following further hypothesis is needed:

(H_1): $Re\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0,\mu=\mu_0} \neq 0$, where τ_0 and μ_0 are bifurcation point and critical frequency, respectively.

Lemma 6 Suppose that $\lambda(\tau) = \gamma(\tau) + i\mu(\tau)$ is the root of Eq. (2) near $\tau = \tau_k$ and satisfying $\gamma(\tau_k) = 0$, $\mu(\tau_k) = \mu_0$, $k = 0, 1, 2, \ldots$, then the following transversality condition

$$Re\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0,\mu=\mu_0} \neq 0 \tag{17}$$

hold.

Proof. Substitute $\lambda(\tau)$ into Eq. (11) and differentiating both sides of it with respect to τ, it can be shown that:

$$P_1'(\lambda)\frac{d\lambda}{d\tau}+P_2'(\lambda)e^{-\lambda\tau}\frac{d\lambda}{d\tau}+P_2(\lambda)e^{-\lambda\tau}\left(-\tau\frac{d\lambda}{d\tau}-\lambda\right)=0,$$

where $P_i'(\lambda)$ are the derivatives of $P_i(\lambda)$ $(i=1,2)$. Hence,

$$\frac{d\lambda}{d\tau}=\frac{M(\lambda)}{N(\lambda)}, \tag{18}$$

where $M(\lambda)=\lambda Ce^{-\lambda\tau}$ and $N(\lambda)=2\alpha\lambda^{2\alpha-1}+A\alpha\lambda^{\alpha-1}-\tau Ce^{-\lambda\tau}$. By straightforward computation, it can be deduced from Eq. (18) that:

$$\frac{d\lambda}{d\tau}=\frac{M_1N_1+M_2N_2+i(M_2N_1-M_1N_2)}{N_1^2+N_2^2},$$

then we take

$$Re\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0,\mu=\mu_0}=\frac{M_1N_1+M_2N_2}{N_1^2+N_2^2},$$

where M_1, M_2 are the real and imaginary parts of $M(\lambda)$ and N_1, N_2 are the real and imaginary parts of $N(\lambda)$, which are defined as

$$M_1=\mu_0C\left[\mu_0^\alpha\cos\frac{\alpha\pi}{2}\sin\mu_0\tau_0-\mu_0^\alpha\sin\frac{\alpha\pi}{2}\cos\mu_0\tau_0\right],$$

$$M_2=\mu_0C\left[\mu_0^\alpha\sin\frac{\alpha\pi}{2}\sin\mu_0\tau_0+\mu_0^\alpha\cos\frac{\alpha\pi}{2}\cos\mu_0\tau_0\right],$$

$$N_1=2a\mu_0^{2\alpha-1}\cos\frac{(2\alpha-1)\pi}{2}+A\alpha\mu_0^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}+ C\left(\alpha\mu_0^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}-\tau_0\mu_0^\alpha\cos\frac{\alpha\pi}{2}\right)\cos\mu_0\tau_0+ C\left(\alpha\mu_0^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}-\tau_0\mu_0^\alpha\sin\frac{\alpha\pi}{2}\right)\sin\mu_0\tau_0,$$

$$N_2=2a\mu_0^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2}+A\alpha\mu_0^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}+ C\left(\alpha\mu_0^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}-\tau_0\mu_0^\alpha\sin\frac{\alpha\pi}{2}\right)\cos\mu_0\tau_0+ C\left(-\alpha\mu_0^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}+\tau_0\mu_0^\alpha\cos\frac{\alpha\pi}{2}\right)\sin\mu_0\tau_0.$$

Thus, from (17), the condition of the occurrence for Hopf bifurcation introduced since

$$\frac{M_1N_1+M_2N_2}{N_1^2+N_2^2}\neq 0. \tag{19}$$

From the above investigation, we can state the following theorem.

Lemma 7. Assume that $(\mathcal{H}_1)$ is satisfied for system (1), the following results hold:

1. The zero equilibrium point is asymptotically stable for $\tau\in[0,\tau_0)$.

2. The system (1) exhibits a Hopf bifurcation at the origin when $\tau=\tau_0$, that is, the system (1) has a branch of periodic solutions bifurcating from the zero equilibrium point near $\tau=\tau_0$.

Now, we study the global stability of the equilibrium points of (1). By considering the linearize system (3) and taking the Laplace transform [19] on both sides of (3), we obtain

$$\begin{aligned} s^\alpha U_1(s)&=-a_1U_1(s)+s^{\alpha-1}\varphi_1(0)-a_2y^*U_1(s)-a_2x^*U_2(s),\\ s^\alpha U_2(s)&=-b_1\left(U_1(s)+\int_{-\tau}^0 e^{-st}\varphi_1(t)\,dt\right)+s^{\alpha-1}\varphi_2(0)-b_2U_2(s),\end{aligned} \tag{20}$$

where $U_1(s)=L(G(t))$ and $U_2(s)=L(I(t))$ are Laplace transform of $G(t)$ and $I(t)$, respectively. Here, it should be mentioned that the initial values $G(t)=\varphi_1(t)$ and $I(t)=\varphi_2(t)$ with $t\in[-\tau,0]$. System (20), can be rewritten as follows:

$$\Delta(s).\begin{bmatrix}U_1(s)\\U_2(s)\end{bmatrix}=\begin{bmatrix}k_1(s)\\k_2(s)\end{bmatrix} \tag{21}$$

in which

$$\Delta(s).\begin{bmatrix}s^\alpha+a_1+a_2y^* & a_2x^*\\ -b_1 & s^\alpha+b_2\end{bmatrix},$$

and

$$k_1(s)=s^{\alpha-1}\varphi_1(0),$$

$$k_2(s)=b_1\int_{-\tau}^{0}e^{-st}\varphi_1(t)\,dt+s^{\alpha-1}\varphi_2(0).$$

Here, $\det\Delta(s)$ as its characteristic polynomial. Thus, the distribution of the eigenvalues of $\det\Delta(s)$ determines the stability of system (9).

Theorem 1. If all the roots of the characteristic equation $\det\Delta(s)=0$ have negative real parts, then the positive equilibrium point E_1 of system (1) is Lyapunov globally asymptotically stable.

Proof. Multiplying both sides of (10) by s gives

$$\Delta(s).\begin{bmatrix}s\,U_1(s)\\s\,U_2(s)\end{bmatrix}=\begin{bmatrix}s\,k_1(s)\\s\,k_2(s)\end{bmatrix}. \tag{22}$$

If all roots of the transcendental equation $\det\Delta(s)=0$ lie in open left complex plane, i.e., $Re\,(s)<0$, then we consider (19) in $Re\,(s)<0$. In this restricted area, system (11) has a unique solution$(sU_1(s),sU_2(s))$, so that:

$$\lim_{\substack{Re\,(s)\geq 0\\ s\to 0}} sU_i(s)=0, i=1,2.$$

From the assumption of all roots of the characteristic Equation

$$\det\Delta(s)=0$$

and the final-value theorem of Laplace transform [19] gives:

$$\lim_{t\to+\infty}G(t)=\lim_{\substack{Re\,(s)\geq 0\\ s\to 0}} sU_1(s)=0,$$

and

$$\lim_{t\to+\infty}I(s)=\lim_{\substack{Re\,(s)\geq 0\\ s\to 0}} sU_2(s)=0,$$

It implies that the zero solution of the fractional-order system (1) is Lyapunov globally asymptotically.

Corollary 1. [20] Suppose that $\tau=0$ and $\alpha\in(0,1)$. If all the roots of the equation

$$\det\Delta(sI-A)=0$$

satisfy

$$|\arg(\lambda)|>\frac{\alpha\pi}{2},$$

then the zero solution of system (4) is Lyapunov globally asymptotically stable.

4. Numerical Simulations

This section deals with some examples and numerical simulations to verify our theoretical results proved in the previous sections by using MATLAB program. We consider the system:

$$\begin{aligned} D^{\alpha}G(t) &= -a_1 G(t) - a_2 G(t)\, I(t) + a_3 \\ D^{\alpha}I(t) &= b_1 G(t-\tau) - b_2\, I(t) \end{aligned} \tag{23}$$

with $a_1 = 0.1335$, $a_2 = 22.04$, $a_3 = 2.333$, $b_1 = 0.22$, $b_2 = 2.200672$. There is a positive equilibrium $E_1 = (0.9992, 0.0999)$ and there is a critical value $\tau_0 = 0.6890$.

Case I. For $\alpha = 1, s = 0.01$, the computer simulations, Figs. (1-4), show that $E_1 = (0.9992, 0.0999)$ is asymptotically stable when $\tau = 0.5090 < \tau_0 = 0.6890$. When τ passes through the critical value $\tau_0 = 0.6890$, E_1 loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from E_1. When $\tau > \tau_0 = 0.6890$, E_1 is unstable, see Figs. (5-6).

Case II. For $\alpha = 0.75, s = 0.01$, the computer simulations, Figs.(7-10), show that E_1 is asymptotically stable when $\tau = 0.5090 < \tau_0 = 0.6890$. When τ passes through the critical value $\tau_0 = 0.6890$, E_1 loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from E_1. When $\tau > \tau_0 = 0.6890$, E_1 is unstable, see Figs. (11-12).

Case III. For $\alpha = 0.99, s = 0.01$, the computer simulations, Figs. (13-16), show that E1 is asymptotically stable when $\tau = 0.4890 < \tau_0 = 0.6890$. When τ passes through the critical value $\tau_0 = 0.6890$, E_1 loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from E_1. When$\tau > \tau_0 = 0.6890$, E_1 is unstable, see Figs. (17-18).

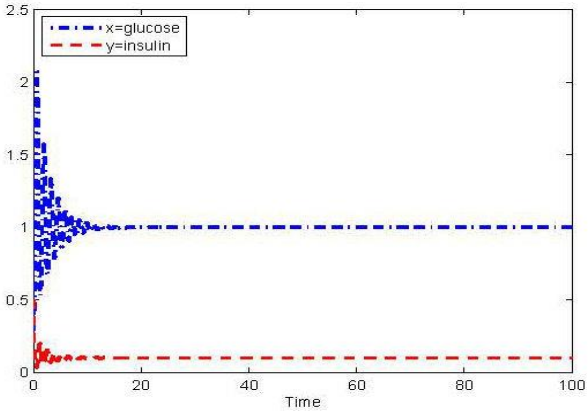

Fig. 1 Phase plain of the Glucose-Insulin dynamics for $\alpha = 1, s = 0.01, \tau = 0.5090$

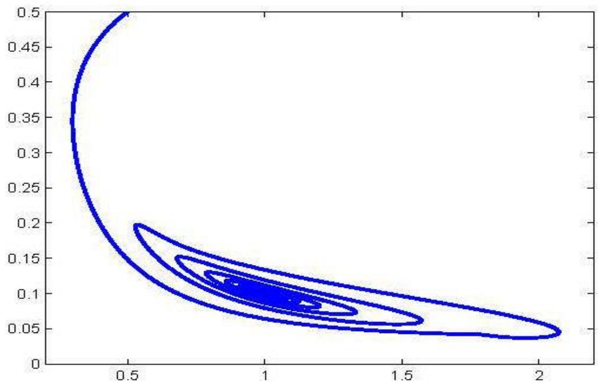

Fig. 2 Glucose-Insulin dynamics for $\alpha = 1, s = 0.01, \tau = 0.5090$

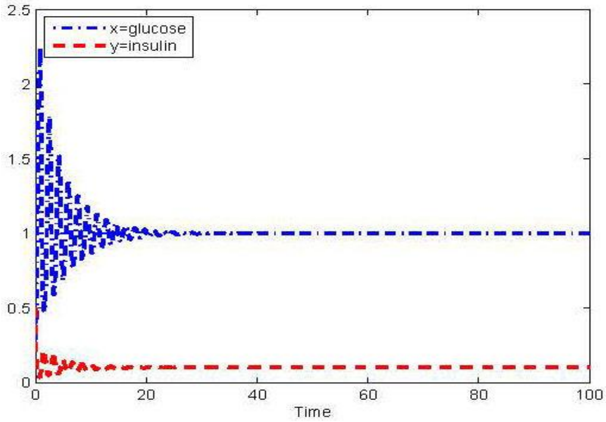

Fig. 3 Phase plain of the Glucose-Insulin dynamics for $\alpha = 1, s = 0.01, \tau = 0.6890$

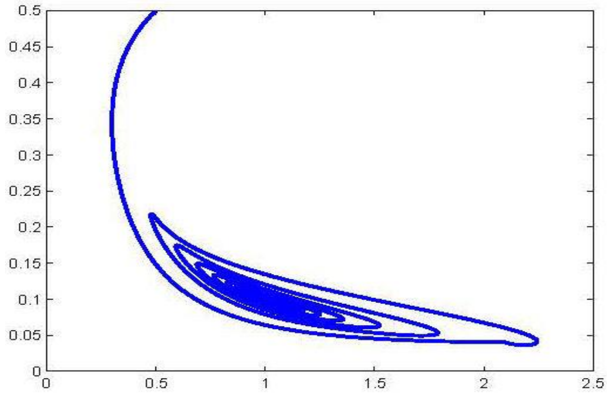

Fig. 4 Glucose-Insulin dynamics for $\alpha = 1, s = 0.01, \tau = 0.6890$

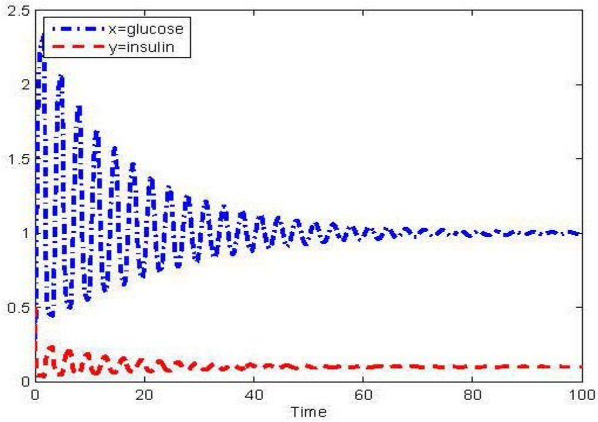

Fig. 5 Phase plain of the Glucose-Insulin dynamics for $\alpha = 1, s = 0.01, \tau = 1.4890$

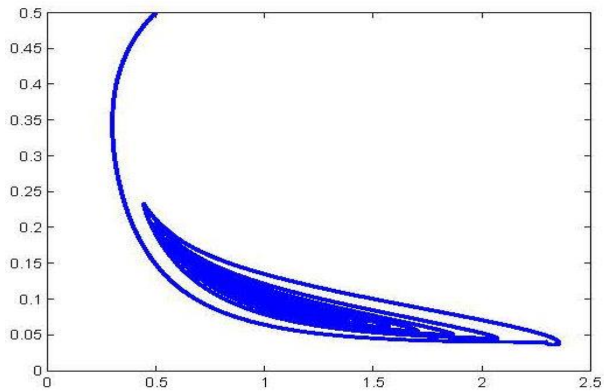

Fig. 6 Glucose-Insulin dynamics for $\alpha = 1, s = 0.01, \tau = 1.4890$

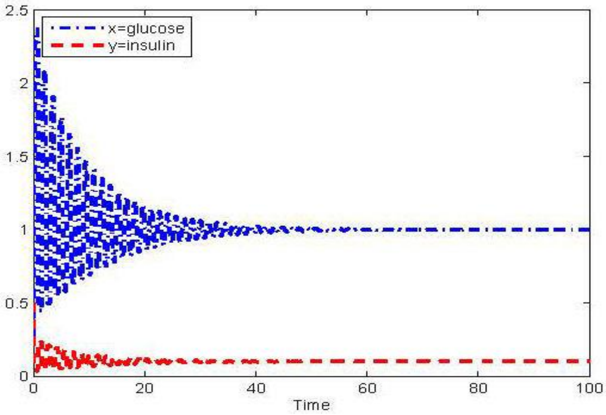

Fig. 7 Phase plain of the Glucose-Insulin dynamics for $\alpha = 0.75, s = 0.01, \tau = 0.6890$

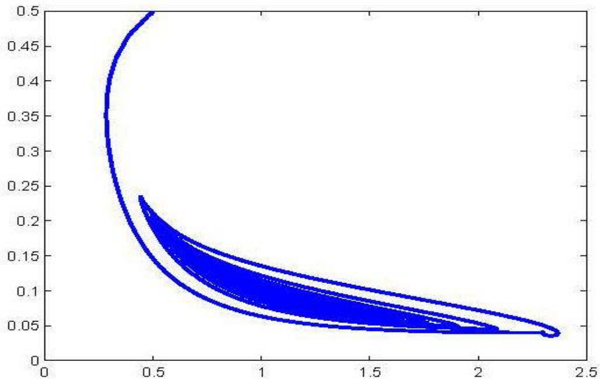

Fig. 8 Glucose-Insulin dynamics for $\alpha = 0.75, s = 0.01, \tau = 0.6890$

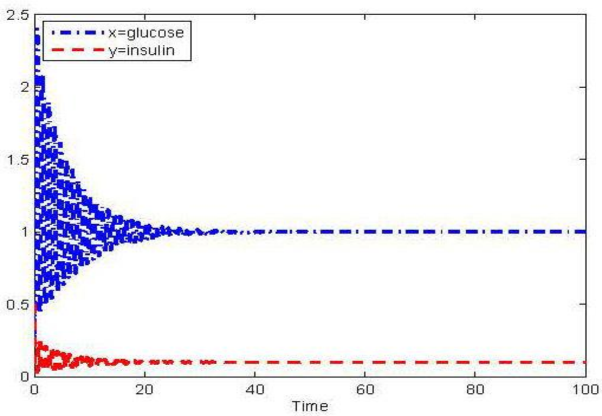

Fig. 9 Phase plain of the Glucose-Insulin dynamics for $\alpha = 0.75, s = 0.01, \tau = 0.5090$

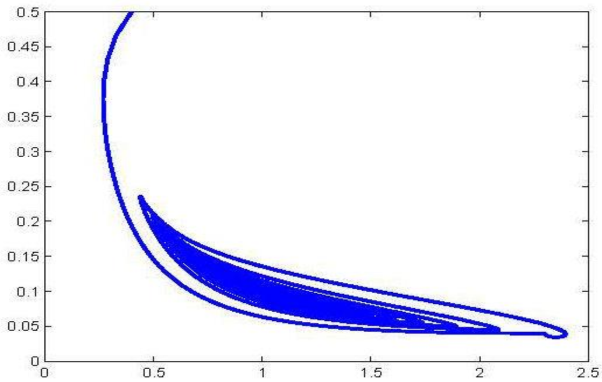

Fig. 10 Glucose-Insulin dynamics for $\alpha = 0.75, s = 0.01, \tau = 0.5090$

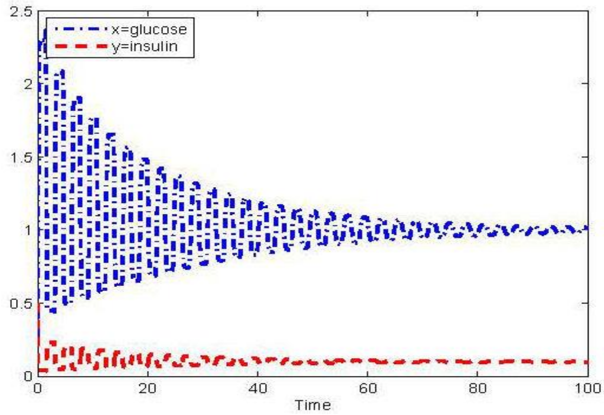

Fig. 11 Phase plain of the Glucose-Insulin dynamics for $\alpha = 0.75, s = 0.01, \tau = 1.4890$

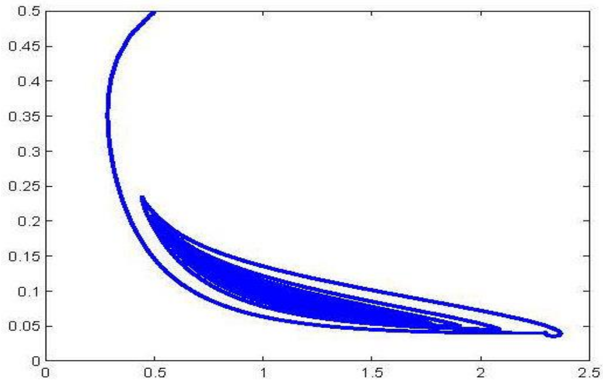

Fig. 12 Glucose-Insulin dynamics for $\alpha = 0.75, s = 0.01, \tau = 1.4890$

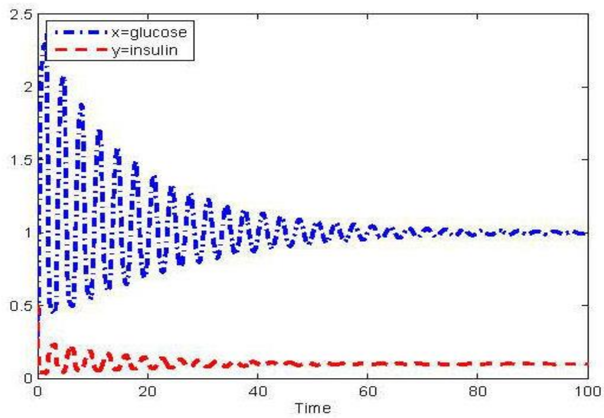

Fig. 13 Phase plain of the Glucose-Insulin dynamics for $\alpha = 0.99, s = 0.01, \tau = 1.4890$

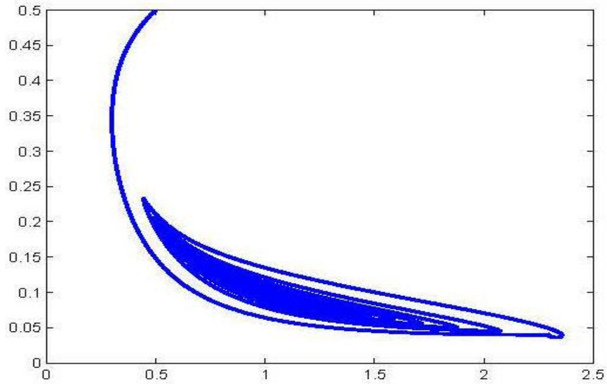

Fig. 14 Glucose-Insulin dynamics for $\alpha = 0.99, s = 0.01, \tau = 1.4890$

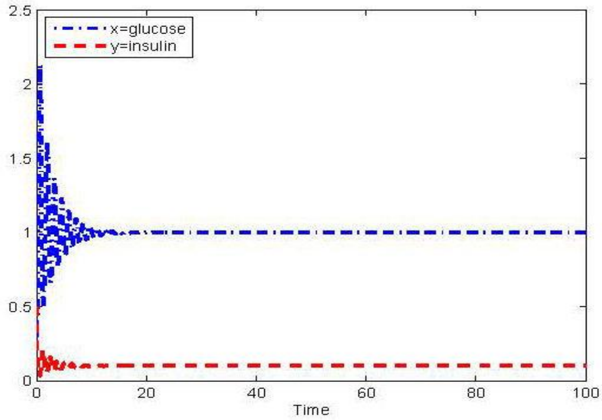

Fig. 15 Phase plain of the Glucose-Insulin dynamics for $\alpha = 0.99, s = 0.01, \tau = 0.4890$

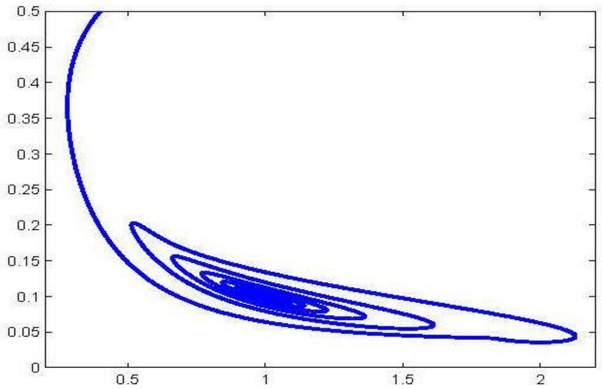

Fig. 16 Glucose-Insulin dynamics for $\alpha = 0.99, s = 0.01, \tau = 0.4890$

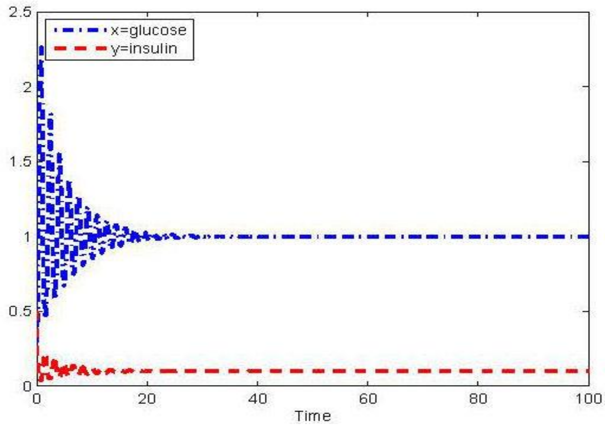

Fig. 17 Phase plain of the Glucose-Insulin dynamics for $\alpha = 0.99, s = 0.01, \tau = .6890$

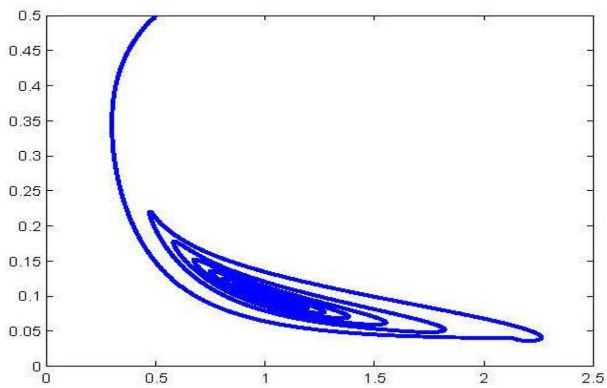

Fig. 18 Glucose-Insulin dynamics for $\alpha = 0.99, s = 0.01, \tau = 0.6890$

2. Conclusion

Delay differential equations are an interesting form of differential equations, with many different applications, particularly in the biological and medical worlds. In this paper, a mathematical model has been proposed and analyzed to study the dynamics of glucose and insulin in the human body. Numerical simulations are carried out to demonstrate the results obtained. Appropriately determining the range of the delay based on physiology and clinical data is important in theoretical study. All the numerical results and graphs presented in the project were in agreement with those presented in the relevant corresponding papers. Our results reveal the conditions on the parameters so that the periodic solution exist surrounding the interior equilibrium. It shows that τ_0 is a critical value for the parameterτ. Furthermore, the direction of Hopf bifurcation and the stability of bifurcated periodic solutions are investigated. From the above results, we conclude that the model is physiologically consistent and may be a useful tool for further research on diabetes.

Acknowledgment

This research is a part of a project entitled "*Different Strategies for Diabetes Diseases in Al-Baha region, Saudi Arabia by Mathematical Modelling*". This project was funded by the Deanship of Scientific Research, Albaha University, KSA (Grant No. 1439/019). The assistance of the deanship is gratefully acknowledged.

References

[1] Saber S, Alzahrani SM, Stability Analysis of a Fractional Order Delayed Glucose-Insulin Model, Albaha University Journal of Basic and Applied Sciences. 2019; 3(2):19–26.

[2] Saber S, Bashier EBM, Alzahrani SM and Noaman IA, A Mathematical Model of Glucose-Insulin Interaction with Time Delay. J Appl Computational Math 2018, Vol 7(3): 416. doi: 10.4172/2168-9679.1000416.

[3] De Gaetano A, Arino O. Mathematical modelling of the intravenous glucose tolerance test. J Math Biol. 2000 Feb; 40(2):136-68.

[4] Hussain J, Zadeng D. A mathematical model of glucose-insulin interaction. Science Vision. 2014; 14(2):84-88.

[5] Oldham KB, Spanier, J. The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.

[6] Miller K, Ross B. An introduction to the fractional calculus and fractional differential equations, Wiley, New York, 1993.

[7] Podlubny I. Fractional differential equations, Academic Press, San Diego, 1999.

[8] Momani S. Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method, Appl. Math. Comput. 2005; 165:459-472.

[9] Odibat Z, Momani S. Numerical solution of Fokker-Planck equation with space- and time-fractional derivatives, Phys. Lett. A. 2007; 369:349-358.

[10] Odibat Z, Momani S. The variational iteration method: An efficient scheme for handling fractional partial differential equations in uid mechanics, Comput. Math. Appl. 2009; 58:2199-2208.

[11] Mustafa I., The approximate and exact solutions of the space and time fractional Burgers equations with initial conditions by variational iteration method, J. Math. Appl. 2008; 345:476-484.

[12] He J-H. Approximate analytical solution for seepage

ow with fractional derivative in porous media, Comput. Methods Appl. Mech. Engrg. 1998; 167:57-68.

[13] Odibat Z, Momani S. Application of variational iteration method to nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul. 2006; 7(1):27-34.

[14] Wu GC, Lee EWM. Fractional variational iteration method and its application, Phys. Lett. A. 2010; 374:2506-2509.

[15] Al-Rabtah A, Erturk VS, Momani S. Solutions of a fractional oscillator by using differential transform method, Comput. Math. Appl. 2010; 59:1356-1362.

[16] Momani S, Odibat Z. Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A. 2007; 365:345-350.

[17] Momani S, Odibat Z. Numerical solutions of the space-time fractional advection-dispersion equation, Numer. Methods Partial Differential Equations. 2008; 24:1416-1429.

[18] Wang Q. Homotopy perturbation method for fractional KdV-Burgers equation, Chaos Solitons Fractals. 2008; 35:843-850.

[19] Deng W, Li C, Lu J. Stability analysis of linear fractional differential system with multiple time delays. Nonlinear Dyn. 2007; 48(4):409-416.

[20] Muth EJ. Transform methods with applications to engineering and operations research. Prentice-Hall, New Jersey, 1977.

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