Stability Analysis of a Fractional Order Delayed Glucose-Insulin Model

Sayed Saber
Department of Mathematics
Faculty of Science and Arts, Baljurashi
Albaha University, Albaha, Saudi Arabia

Salem Mubarak Alzahrani
Department of Mathematics
Faculty of Arts and Science in Almandaq
Albaha University, Albaha, Saudi Arabia
CONTENTS

Research

1 Analysis of Mobile Malwares Attacks Using Deep Learning Classification
Mohammad Eid Alzahrani

7 Hematological Indices of Pregnant Sudanese Woman Attended Wad Medani Health Care Centers in Gezira State, Sudan
Algurashi A. Abuelgasim, Hajir Mohammed Hussien Omer, Khalid Eltahir Khalid, Abd Elrahim Haggaz

11 Preparation of Economic Belite Cement from Saudi Raw Materials
Abdulaziz Ali Alomari

19 Stability Analysis of a Fractional Order Delayed Glucose-Insulin Model
Sayed Saber, Salem Mubarak Alzahrani

27 Hopf Bifurcation on Fractional Ordered Glucose-Insulin System with Time-Delay
Sayed Saber, Salem Mubarak Alzahrani

Author guidelines

35 Author Guidelines
Stability Analysis of a Fractional Order Delayed Glucose-Insulin Model

Sayed Saber, Salem Mubarak Alzahrani

ABSTRACT

In this paper, we investigate the stability analysis of a fractional order delayed glucose-insulin model. The equilibrium points are computed and stability of the equilibrium points are analyzed. Local and global stability of existence steady states and Hopf bifurcation with respect to the delay is investigated, with fractional order \( \alpha \in (0,1) \). The phase portraits are obtained for different sets of parameter values. Numerical simulations are performed and it is shown that the system exhibits rich dynamical behaviors.

1. Introduction

Diabetes Mellitus is a disease, which characterized by too high sugar levels in the blood and urine. Many mathematical models for glucose metabolism have been proposed, and they range in complexity from a set of two coupled ordinary linear differential equations of the first order with constant coefficients [1] to an elaborate analog simulation [2]. Among the most widely used models to study diabetes dynamics (see for example [3, 4, 5, 6, 7]), is the minimal model which is used by Hussain-Zadeng [8], reported a model for the interaction of glucose and insulin model built on the work of [8] as follows:

\[
\begin{align*}
Dx(t) &= -a_1 x(t) - a_2 x(t - \tau) y(t - \tau) + a_3, t \in [0,T], \\
Dy(t) &= b_1 x(t) - b_2 y(t), t \in [0,T],
\end{align*}
\]

where \( x \geq 0, \ y \geq 0 \), represents glucose and insulin concentration respectively, \( a_1 \) and \( a_2 \) are the rate constant which represents insulin-independent and insulin-dependent glucose disappearance respectively, \( a_3 \) is the glucose infusion rate and \( b_1 \) is the rate constant which represents insulin production due to glucose stimulation, \( b_2 \) is the rate constant which represents insulin degradation.

In [9], the authors extended the model (1) by incorporating a time delay in the interaction term of the glucose and insulin. They studied the following model:

\[
\begin{align*}
Dx(t) &= -a_1 x(t) - a_2 x(t - \tau) y(t - \tau) + a_3, t \in [0,T], \\
Dy(t) &= b_1 x(t) - b_2 y(t), t \in [0,T],
\end{align*}
\]

where the time delay \( \tau \) represents the time taken by Pancreas to respond to the feedback of the glucose level and it is used as the bifurcation parameter.

In recent years, fractional order differential equations have become an important tool in mathematical modeling [10]. The use of fractional-order differential and integral operators in mathematical models has become increasingly widespread in recent years [11, 12].

In this paper, we analyses the dynamical behavior of the following fractional order, which takes time delay into consideration, and is described as:

\[
\begin{align*}
D^\alpha x(t) &= -a_1 x(t) - a_2 x(t - \tau) y(t - \tau) + a_3, t \in [0,T], \\
D^\alpha y(t) &= b_1 x(t) - b_2 y(t), \ t \in [0,T],
\end{align*}
\]

with initial data:

\[
\begin{align*}
x(\theta) &= \phi(\theta), \quad \theta < 0, \\
y(\theta) &= \psi(\theta), \quad \theta < 0.
\end{align*}
\]

System (3), is the discrete version delay fractional order of the system (2). Here, we show that Laplace transform can be applied to fractional system. More precisely, the stability of equilibrium points is studied. In addition, we provide theoretical analysis, using the eigenvalues method and
linearization techniques and bifurcation method. Numerical simulations have been used to verify the theoretical analysis.

2. Preliminary

There are three main definitions of fractional-order differential, that is, Grunwald-Letnikov, Riemann-Liouville and Caputo’s definitions. Here, this paper is based on the definition of Caputo.

Definition 1. [11] The Caputo fractional derivative of order \( \alpha > 0, n - 1 < \alpha < n, n \in \mathbb{N} \) is defined as:

\[
D^\alpha f(t) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^\alpha} dx, & n - 1 < \alpha < n, \\
d^n f(t), & \alpha = n,
\end{array} \right.
\]

where \( \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \) is the Euler gamma function.

Lemma 1. [13] The equilibrium point \( E_1 = (x^*, y^*) \) of the fractional differential system:

\[
\begin{align*}
D^\alpha x(t) &= f_1(x,y), & x(0) &= x_0, \\
D^\alpha y(t) &= f_2(x,y), & y(0) &= y_0,
\end{align*}
\]

is locally asymptotically stable if and only if all eigenvalues \( \lambda_i \) of the Jacobian matrix:

\[
J = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix}
\]

evaluated at the equilibrium point \( E_1 \), satisfy the condition that \( \arg(\lambda_i) > -\frac{\alpha \pi}{2} \).

Lemma 2. [14] Considering the fractional differential system with Caputo derivative:

\[
D^\alpha X = AX, \quad X(0) = X_0,
\]

with \( \alpha \in (0,1], X \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{nxn} \). The characteristic equation of the system is \( \text{det}[s^\alpha I - A] = 0 \). If all of the roots of the characteristic equation have negative real parts, then the zero solution of the system is asymptotically stable.

Lemma 3. [15] Considering the fractional delayed differential system with Caputo derivative:

\[
D^\alpha X(t) = AX(t) + B X(t - \tau), X(t) = \Phi(t), t \in [-\tau, 0]
\]

with \( \alpha \in (0,1], X \in \mathbb{R}^n, \tau \in \mathbb{R}^{nxn} \) and \( A \in \mathbb{R}^{nxn} \). The characteristic equation of the system is \( \text{det}[s^\alpha I - A - B e^{-s\tau}] = 0 \). If all of the roots of the characteristic equation have negative real parts, then the zero solution of the system is asymptotically stable.

Theorem 1. [15] The zero solution of system (3) is Lyapunov globally asymptotically stable if all the roots of the characteristic equation of (3) have negative real parts.

Remark 1. If \( \alpha = 1 \), the characteristic matrix and characteristic equation of (3) are reduced to \( \text{det}(sI - A) = 0 \) and \( \text{det}(sI - A) = 0 \), respectively. If \( \tau = 0 \), the characteristic matrix and characteristic equation of (3) are respectively reduced to \( sI - A \) and \( \text{det}(sI - A) = 0 \).

Theorem 2. [16] If \( \alpha = 1 \), a set of necessary and sufficient conditions for \( E_1 \) to be asymptotically stable for all \( \tau \geq 0 \) is the following:

1. The real parts of all the roots of \( \Delta(\lambda, 0) = 0 \) are negative.
2. For all real \( \omega \) and \( \tau \geq 0, \Delta(\omega, \tau) \neq 0, i = \sqrt{-1} \).

3. Local stability analysis and Hopf bifurcation

In this Section, we propose the conditions of the system (3) to undergo a Hopf bifurcation at the equilibrium \( E_1 \) when \( \tau = \tau_0 \) as follows:

\[
\begin{align*}
(\text{H1}) & \quad \text{All the eigenvalues of the coefficient matrix of the linearized system of (3) satisfy } |\arg(\lambda_i)| > \frac{\alpha \pi}{2}. \\
(\text{H2}) & \quad \text{The characteristic equation of (3) has a purely imaginary roots } \pm i\omega_0 \text{ when } \tau = \tau_0. \\
(\text{H3}) & \quad \text{Re}\left[ \frac{d\lambda}{d\tau} \right]_{\tau=\tau_0,\omega=\omega_0} \neq 0, \text{ where } \text{Re} \text{ denotes the real part of the complex eigenvalue.}
\end{align*}
\]

The steady state or equilibrium (fixed point) of the system (3) is one of which

\[
\begin{align*}
x(t) &= x(t - \tau) \text{ and } D^\alpha x(t) = 0, \\
y(t) &= y(t - \tau) \text{ and } D^\alpha y(t) = 0.
\end{align*}
\]

It is easy to see that system (3) has the equilibrium point \( E_1 = (x^*, y^*) \) as the following:

\[
\begin{align*}
x^* &= -a_1b_2 + \sqrt{(a_1b_2)^2 + 4a_2b_2a_3b_1}, \\
y^* &= -a_1b_2 - \sqrt{(a_1b_2)^2 + 4a_2b_2a_3b_1}.
\end{align*}
\]

The interior-equilibrium point \( E_1 = (x^*, y^*) \) exists unconditionally as \( x^* \) and \( y^* \) are always positive as all the parameters are considered positive. To linearize the model (3) about the equilibrium point \( E_1 \), let \( u_1(t) = x(t) - x^* \), \( u_2(t) = y(t) - y^* \). After removing nonlinear terms, one obtains the linear variational system:

\[
D^\alpha u_1(t) = -a_1u_1(t) - a_2y^*u_1(t - \tau) - a_3x^*u_2(t - \tau),
\]

\[
D^\alpha u_2(t) = -b_1u_1(t) - b_2u_2(t), 0 < \alpha \leq 1.
\]

Taking Laplace transform [17] on both sides of (4), one obtains the associated characteristic equation as follows:

\[
\left| \begin{array}{cc}
2^{\alpha a} + a_1 + a_2 y^* e^{-\lambda \tau} & a_1 x^* e^{-\lambda \tau} \\
-b_1 & a_2 x^* e^{-\lambda \tau} + b_2
\end{array} \right| = 0.
\]

Thus, the characteristic equation is:

\[
\lambda^2 + a_1 + a_2 y^* + c + (b_2\lambda^2 + d)e^{-\lambda \tau} = 0,
\]

where \( a = a_1 + b_2, b = a_2 y^*, c = a_1 b_2 \) and \( d = a_2 b_1 x^* + a_2 b_2 y^* \).

3.1. The case \( \alpha = 1 \)

Lemma 4. [9] For \( \alpha = 1 \) with \( \tau = 0 \), the positive equilibrium point \( E_1 \) of system (4) is locally asymptotically stable if and only if both conditions

\[
a + b > 0 \text{ and } c + d > 0
\]

(6)
hold simultaneously.

**Proof.** If \( \alpha = 1 \), Eq. (5) becomes

\[
\Delta(\lambda, \tau) = \lambda^2 + a\lambda + c + (b\lambda + d)^{-\alpha}.
\]

(7)

For \( \tau = 0 \), Eq. (7) becomes

\[
\lambda^2 + (a + b)\lambda + c + d = 0.
\]

(8)

The sum of the roots is \( -(a + b) \) and the product of the roots is \( c + d \). Thus, one can say that both the roots of (8) are real and negative or complex conjugate with negative real parts if and only if (6) hold simultaneously. Hence, in the absence of time delay, the equilibrium point \( E_1 \) is locally asymptotically stable if and only if (6) hold simultaneously.

**Lemma 5.** \((9)\) For \( \tau \neq 0 \), if conditions (6) and if

\[
a^2 - b^2 - 2c > 0 \text{ and } c^2 - d^2 > 0
\]

(9)

are satisfied, then the equilibrium \( E_1 \) is asymptotically stable for all \( \tau < \tau_0 \) and unstable for \( \tau > \tau_0 \). Furthermore, as \( \tau \) increase through \( \tau_0 \), \( E_1 \) bifurcates into small amplitude periodic solutions, where \( \tau_0 = \tau_0n = n \pi \).

**Proof.** For \( \tau \neq 0 \), if \( \lambda = \iota\omega(\omega > 0) \) is a root for the characteristic equation (5), then \( \omega \) should satisfy the following equations

\[
\omega b \sin \omega \tau + d \sin \omega \tau = \omega^2 - c,
\]

(10)

\[
\omega b \cos \omega \tau - d \cos \omega \tau = -\omega a
\]

which implies that

\[
\omega^4 + (a^2 - b^2 - 2c)\omega^2 + c^2 - d^2 = 0.
\]

(11)

Then, one obtains:

\[
\omega^2_0 = \left(\frac{-a^2 - b^2 - 2c \pm \sqrt{(a^2 - b^2 - 2c)^2 - 4(c^2 - d^2)}}{2}\right)^{1/2}
\]

From (10), it follows that if \( \iota \) holds then Eq. (11) does not have any real solutions.

From (10), we see that there is a unique positive solution \( \omega^2_0 \) if \( c^2 - d^2 < 0 \). If \( c^2 - d^2 > 0 \), \( a^2 - b^2 - 2c > 0 \) and \( a^2 - b^2 - 2c > 4(c^2 - d^2) \) hold, then there are two positive solutions \( \omega^2_{01} \) and \( \omega^2_{02} \) into (10). Substituting in (10) and solving for \( \tau \), we get

\[
\tau_0 = \frac{1}{\omega_0} \arctan \left( \frac{\omega_0(ad - bc + b\omega^2_0)}{ab\omega^2_0 + (c - \omega_0^2)d} \right) + \frac{2\pi n}{\omega_0}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

Substituting \( \omega^2_0 \) into (10) and solving for \( \tau \) one obtains

\[
\tau^* \equiv \frac{1}{\omega^2_{0}} \arctan \left( \frac{\omega^2_{0}(ad - bc + b\omega^2_{0})}{ab\omega^2_{0} + (c - \omega^2_{0})d} \right) + \frac{2\pi n}{\omega^2_{0}}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

Differentiating Eq. (5) with respect to \( \tau \), one obtains

\[
[2\lambda + a + be^{-\lambda\tau} - \tau(b\lambda + d)e^{-\lambda\tau}] \frac{d\lambda}{d\tau} = \lambda e^{-\lambda\tau}(b\lambda + d).
\]

Thus

\[
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + a}{-\lambda^2 + \alpha a + c} + \frac{b}{b\lambda + d} - \frac{\tau}{\lambda}.
\]

and by using

\[
e^{-\lambda\tau} = \left( \frac{\lambda^2 + \alpha a + c}{b\lambda + d} \right),
\]

one obtains

\[
sign \left( \frac{d(\text{Re} \lambda)}{d\tau} \right)_{\lambda = \iota \omega} = \text{sign} \left( \text{Re} \left( \frac{d\lambda}{d\tau} \right)^{-1} \right)_{\lambda = \iota \omega} \]

\[
sign \left( \frac{d(\text{Re} \lambda)}{d\tau} \right)_{\lambda = \iota \omega} = \frac{2a(\alpha^{-1}c^2 + \alpha c^2)}{b^2 + \alpha b^2 d^2}.
\]

(12)

Hence, by Butler’s lemma, \( E_1 \) remains stable for \( \tau < \tau_0 \). Now, we need to show that

\[
\frac{d(\text{Re} \lambda)}{d\tau} \bigg|_{\tau = \tau_0, \omega = \omega_0} > 0.
\]

This will signify that there exists at least one eigenvalue with positive real part for \( \tau > \tau_0 \). Moreover, the conditions of Hopf bifurcation [18] are then satisfied yielding the required periodic solution. It follows, from (12), that

\[
sign \left( \frac{d(\text{Re} \lambda)}{d\tau} \right)_{\lambda = \iota \omega} = \text{sign} \left( \frac{2a(\alpha^{-1}c^2 + \alpha c^2)}{b^2 + \alpha b^2 d^2} \right)
\]

Thus

\[
\frac{d(\text{Re} \lambda)}{d\tau} \bigg|_{\tau = \tau_0, \omega = \omega_0} > 0.
\]

Thus, the transversality condition holds, and hence, Hopf bifurcation occurs at \( \tau = \tau_0 \). Thus, the proof follows.

**3.2. The case \( \alpha \in (0, 1] \)**

Eq. (5) can be rewritten equivalently as:

\[
P_1(\lambda) + P_2(\lambda)e^{-\lambda\tau} = 0,
\]

(13)

\[
P_1(\lambda) = \lambda^{2\alpha} + a\lambda^\alpha + c, \quad P_2(\lambda) = b\lambda^\alpha + d.
\]

Assume that \( \lambda = \iota \omega = \omega \left( \cos \frac{\pi}{2} + \iota \sin \frac{\pi}{2} \right) \) is a root of Eq. (5), \( \omega > 0 \). Substituting \( \lambda \) into Eq. (5) and separating the real and imaginary parts of it, it results in:

\[
A_2 \cos \omega \tau + B_2 \sin \omega \tau = -A_1,
\]

\[
B_2 \cos \omega \tau - A_2 \sin \omega \tau = -B_1,
\]

(14)

where

\[
A_1 = \omega^2 a \cos \alpha \omega + a \omega^\alpha \cos \alpha \omega^2 + c,
\]

\[
A_2 = \omega \left[ b \omega^\alpha \sin \frac{\pi}{2} + d \right] \sin \omega \tau + \left[ b \omega^\alpha \cos \frac{\pi}{2} + d \right] \cos \omega \tau,
\]

\[
B_1 = 2a \omega^2 a \cos \frac{2(\alpha - 1)\pi}{2} + a \omega^\alpha \sin \frac{2\alpha \pi}{2} + b(\alpha \omega^\alpha \cos \frac{2(\alpha - 1)\pi}{2} - \tau b \omega^\alpha \cos \frac{2\alpha \pi}{2} - \tau d) \cos \omega \tau + b(\alpha \omega^\alpha \sin \frac{2(\alpha - 1)\pi}{2} + \tau b \omega^\alpha \sin \frac{2\alpha \pi}{2} - \tau d) \sin \omega \tau,
\]

\[
B_2 = 2a \omega^2 a \sin \frac{2(\alpha - 1)\pi}{2} + a \omega^\alpha \sin \frac{2\alpha \pi}{2} + b(\alpha \omega^\alpha \sin \frac{2(\alpha - 1)\pi}{2} - \tau b \omega^\alpha \sin \frac{2\alpha \pi}{2} + \tau d) \cos \omega \tau + (b \alpha \omega^\alpha \cos \frac{2(\alpha - 1)\pi}{2} + \tau b \omega^\alpha \cos \frac{2\alpha \pi}{2} + \tau d) \sin \omega \tau.
\]

Utilizing Eq. (14), direct calculation yields:

\[
\cos \omega \tau = A_1 A_2 + B_1 B_2
\]

\[
\sin \omega \tau = A_2 B_1 - A_1 B_2
\]

It is clear that
Thus, it follows from \( \cos \omega t = G_1(\omega) \) that
\[
\tau^{(k)} = \frac{1}{\omega} \arccos \frac{G_1(\omega)}{\omega}, k = 0, 1, 2, \ldots.
\] (15)

We suppose that Eq. (14) has at least one positive real root. Define the bifurcation point:
\[
\tau_0 = \min(\tau^{(k)}), k = 0, 1, 2, \ldots
\]
where \( \tau^{(k)} \) is defined by (15).

**Lemma 6.** Suppose that \( \lambda(\tau) = \gamma(\tau) + i\omega(\tau) \) denotes the root of Eq. (5) near \( \tau = \tau_k \) satisfying \( \gamma(\tau_k) = 0, \omega(\tau_k) = \omega_0, k = 0, 1, 2, \ldots \) then the following transversality condition holds:
\[
\text{Re} \left[ \frac{d\lambda}{d\tau} \right]_{\tau=\tau_0, \omega=\omega_0} \neq 0.
\] (16)

**Proof.** To derive the condition of the occurrence for Hopf bifurcation, we introduce the following hypothesis:
\[
\frac{M_1 N_1 + M_2 N_2}{N_1^2 + N_2^2} \neq 0,
\] (17)
where
\[
M_1 = \omega \left[ \left( b a^s \cos \frac{\alpha \pi}{2} + d \right) \sin \omega t - \left( b a^s \sin \frac{\alpha \pi}{2} \cos \omega t \right) \right],
\]
\[
M_2 = \omega \left[ \left( b a^s \cos \frac{\alpha \pi}{2} + d \right) \sin \omega t + \left( b a^s \cos \frac{\alpha \pi}{2} \cos \omega t \right) \right],
\]
\[
N_1 = 2 a a^s a^{-1} \cos \left( \frac{2a a^s a^{-1} \pi}{2} + a a^s a^{-1} \cos \frac{2a a^s a^{-1} \pi}{2} + a a^s a^{-1} \cos \frac{2a a^s a^{-1} \pi}{2} + a a^s a^{-1} \cos \frac{2a a^s a^{-1} \pi}{2} \right),
\]
\[
N_2 = 2 a a^s a^{-1} \sin \left( \frac{2a a^s a^{-1} \pi}{2} + a a^s a^{-1} \cos \frac{2a a^s a^{-1} \pi}{2} + a a^s a^{-1} \cos \frac{2a a^s a^{-1} \pi}{2} + a a^s a^{-1} \cos \frac{2a a^s a^{-1} \pi}{2} \right) \sin \omega t.
\]

Substitute \( \lambda(t) \) into (5) and differentiating both sides of it with respect to \( \tau \), we can achieve that:
\[
P_1(\lambda) \frac{d\lambda}{d\tau} + P_2(\lambda) e^{-\lambda \tau} \frac{d\lambda}{d\tau} + P_3(\lambda) e^{-\lambda \tau} \left( -\frac{d\lambda}{d\tau} - \lambda \right) = 0,
\]
where \( P_i(\lambda) \) are the derivatives of \( P_i(\lambda) \) (\( i = 1, 2 \)). Hence,
\[
\frac{d\lambda}{d\tau} = M(\lambda)
\]
where
\[
M(\lambda) = \lambda (b \lambda^s + d) e^{-\lambda \tau} \text{ and } N(\lambda) = 2 a a^s a^{-1} + a a^s a^{-1} + (b a a^s a^{-1} - b \lambda^s a - \tau d) e^{-\lambda \tau}.
\]

By straightforward computation, it can be deduced from Eq. (13) that:
\[
\text{Re} \left[ \frac{d\lambda}{d\tau} \right]_{\tau=\tau_0, \omega=\omega_0} = \frac{M_1 N_1 + M_2 N_2}{N_1^2 + N_2^2}.
\]
The Eq. (16) follows by using Eq. (17).

**Remark 2.** Lemma 6 implies that the transversality condition (16) of Hopf bifurcations is satisfied for the delayed fractional-order model (3).

**Remark 3.** (i) The zero equilibrium point is asymptotically stable for \( \tau = 0 \).

(ii) System (3) undergoes a Hopf bifurcation at the origin when \( \tau = 0 \).

**Remark 4.** Lemma 5 implies that the transversality condition (9) of Hopf bifurcations is satisfied for the delayed fractional-order model (3).

As in [19], one obtains the following result:

**Lemma 7.** The equilibrium point of model (3) is asymptotically stable for \( \tau \in (0, \tau_0) \), and unstable when \( \tau > \tau_0 \).

**Proof.** Note that the coefficient matrix of the linearized (4) has the eigenvalue \( \lambda < 0 \). Thus, the condition (H1) of Hopf bifurcations is satisfied for model (3). It is easy to see that all the roots of (5) with \( \tau = 0 \) have negative real parts. From (15), \( \tau_0 \) implies that all the roots of (5) have negative real parts for \( \tau \in [0, \tau_0) \). The conclusion in Lemma 6 indicates that (5) has at least one root with positive real part when \( \tau = \tau_0 \). Thus, the conclusion follows.

**Lemma 8.** Model (3) undergoes a Hopf bifurcation at the equilibrium point when \( \tau = \tau_0 \).

**Proof.** From Remarks 2 and 4, we know that the conditions (9) and (16) of Hopf bifurcations are satisfied for model (3). Hence, a Hopf bifurcation occurs at the equilibrium point when \( \tau = \tau_0 \).

4. Global Stability Analysis

In this section, we investigate the stability and bifurcation of the delayed fractional-order model (3) of glucose-insulin interaction. To study the global stability of the equilibrium points of (4), we consider the linearize system (4), Taking the Laplace transform [17] on both sides of (4) gives:
\[
s^a U_1(s) = -a_1 U_1(s) + s^{a-1} \varphi_1(0)
\]
\[
- a_2 y^* \left[ U_1(s) + \int_0^\tau e^{-\tau} \varphi_1(t) dt \right]
\]
\[
- a_2 x^* \left[ U_2(s) + \int_0^\tau e^{-\tau} \varphi_2(t) dt \right].
\]
\[
s^a U_1(s) = b_1 U_1(s) + s^{a-1} \varphi_1(0) - b_2 U_2(s),
\] (17)
where \( U_1(s) = L(u_1(t)) \) and \( U_2(s) = L(u_2(t)) \) are Laplace transform of \( u_1(t) \) and \( u_2(t) \), respectively. Here, it should be mentioned that the initial values \( u_1(t) = \varphi_1(t) \) and \( u_2(t) = \varphi_2(t) \) with \( t \in [-\tau, 0] \). The system (17) can be rewritten as follows:
\[
\begin{bmatrix}
U_1(s)
\end{bmatrix}
= k_1(s)
\begin{bmatrix}
U_2(s)
\end{bmatrix}
= k_2(s),
\] (18)
in which
\[
\Delta(s) = \begin{bmatrix}
 s^a + a_1 + a_2 y^* & a_2 x^*
\end{bmatrix}.
\]

and
\[
k_1(s) = s^{a-1} \varphi_1(0) - a_2 y^* \int_0^\tau e^{-\tau} \varphi_1(t) dt - a_2 x^* \int_0^\tau e^{-\tau} \varphi_2(t) dt,
\]
\[
k_2(s) = s^{a-1} \varphi_2(0).
\]
Here, \( \Delta(s) \) is considered as a characteristic matrix of system (4) and \( \det \Delta(s) \) as its characteristic polynomial. Thus, the distribution of the eigenvalues of \( \det \Delta(s) \) determines the stability of the system (4).

**Theorem 3.** If all the roots of the characteristic equation \( \det \Delta(s) = 0 \) have negative real parts, then the positive equilibrium point \( E_1 \) of system (4) is Lyapunov globally asymptotically stable.

**Proof.** \( \Delta(s) \) is a characteristic matrix of system (4), so the roots of characteristic equation \( \det \Delta(s) = 0 \) are the eigenvalues of \( \Delta(s) \). According to the Routh-Hurwitz criterion, if all the roots of the characteristic equation \( \det \Delta(s) = 0 \) have negative real parts, then the positive equilibrium point \( E_1 \) of system (4) is Lyapunov globally asymptotically stable.

**Example 2.** We consider the system:

\[
D^\alpha x(t) = -0.1135x(t) - 1.4x(t - \tau)y(t - \tau) + 1.23, \\
D^\alpha y(t) = 0.22x(t) - 0.2972y(t).
\]

When \( \alpha = 0.99 \), we choose \( \tau = 1.0434 < \tau_0 = 1.2434 \), which is the same value as that used in Fig. 7. According to Theorem 3, we conclude that instead of having a Hopf bifurcation, the fractional-order system (4) with \( \alpha = 0.99 \) converges to the equilibrium point \( E_1 \), as shown in Fig. 7, Fig. 8, Fig. 9 and Fig. 10, that the critical value \( \alpha = 0.99 \) increases from 1.2434 to 1.5434, implying that the onset of Hopf bifurcations is delayed, as shown in Figs. (11 and 12).

**Remark.** It can be shown that if we choose a smaller value of \( \alpha \), then the fractional-order model (4) may not have a Hopf bifurcation even for the larger values of \( \tau \). This indicates that the order \( \alpha \) can delay the onset of Hopf bifurcations, thus guaranteeing a stationary sending rate for the larger values of \( \tau \).

6. Conclusion

In this paper, we have extended a delayed glucose-insulin model to a fractional order counterpart. We have considered the stability of linear fractional glucose-insulin interaction systems with time delay. We discovered that if all roots of the characteristic equation have negative parts, then the equilibrium of the above linear system with fractional order is Lyapunov globally asymptotically stable. Based on this introduced characteristic equation, several interesting stability criteria are derived. As an application, we apply our results to the delayed system and determine the asymptotically stable region of the system. Using these obtained results, we successfully determine a sufficient stability condition for a delayed fractional differential equation.

We have also proposed some conditions of Hopf-type bifurcations for delayed fractional-order systems. By using the Laplace transform, we introduced a characteristic equation for the system with time delay. Numerical experiments are performed for different values of the derivative and the time-delay term. It is observed that chaotic system gets stabilized for some values of delay.

Acknowledgment

This research is a part of a project entitled “Different strategies for Diabetes diseases in Al-Baha region, Saudi Arabia by Mathematical modelling”. This project was funded by the Deanship of Scientific Research, Alba University, KSA (Grant No. 1439/019). The assistance of the deanship is gratefully acknowledged.
Fig. 1 Glucose-Insulin dynamics for $\tau = 1.0434$, $\alpha = 1$, $s = 0.01$.

Fig. 2 Phase plane of the Glucose-Insulin dynamics for $\tau = 1.0434$, $\alpha = 1$, $s = 0.01$.

Fig. 3 Glucose-Insulin dynamics for $\tau = 1.2434$, $\alpha = 1$, $s = 0.01$.

Fig. 4 Phase plane of the Glucose-Insulin dynamics for $\tau = 1.2434$, $\alpha = 1$, $s = 0.01$.

Fig. 5 Glucose-Insulin dynamics for $\tau = 1.5434$, $\alpha = 1$, $s = 0.01$.

Fig. 6 Phase plane of the Glucose-Insulin dynamics for $\tau = 1.5434$, $\alpha = 1$, $s = 0.01$. 
Fig. 7 Glucose-Insulin dynamics for \( \tau = 1.0434 \), \( \alpha = 0.99 \), \( s = 0.01 \).

Fig. 8 Phase plain of the Glucose-Insulin dynamics for \( \tau = 1.0434 \), \( \alpha = 0.99 \), \( s = 0.01 \).

Fig. 9 Glucose-Insulin dynamics for \( \tau = 1.2434 \), \( \alpha = 0.99 \), \( s = 0.01 \).

Fig. 10 Phase plain of the Glucose-Insulin dynamics for \( \tau = 1.2434 \), \( \alpha = 0.99 \), \( s = 0.01 \).

Fig. 11 Glucose-Insulin dynamics for \( \tau = 1.5434 \), \( \alpha = 0.99 \), \( s = 0.01 \).

Fig. 12 Phase plain of the Glucose-Insulin dynamics for \( \tau = 1.5434 \), \( \alpha = 0.99 \), \( s = 0.01 \).
References
